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On an Inequality Related to the Isotonicity of the Projection Operator

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In connection to the study of the isotonicity of the projection operator onto a closed convex set in an ordered Hilbert space, Isac has recently remarked the importance of an inequality named "the property of four elements" (PFE). In this paper a sharp inequality closely connected to (PFE) is proved in a Banach space setting. The property $(\text{PFE})_V$ for Lyapunov functionals V is introduced and studied. Some applications are included.

1. INTRODUCTION

The metric projection operator P_D onto a closed convex set D in a Hilbert space or in a Banach space (satisfying some special properties) has been deeply investigated and applied in different areas of mathematics such as functional and numerical analysis, optimization and complementarity theory etc. (see e.g. [1, 4, 6, 7–14, 17, and 20–21]). The projection operator has been studied from several points of view as for example: smoothness, differentiability, uniform continuity, etc. Another important and interesting property seems to be the isotonicity with respect to an ordering defined by a closed convex cone.

If $(H, <, >)$ is a Hilbert space ordered by a closed convex cone K, the ordering defined by K is that $x \leqslant y$ if and only if $y - x \in K$.

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If $D \subset H$ is a closed convex set the projection operator P_D onto D is defined by

$$\|P_D(x) - x\| = \inf_{y \in D} \|y - x\|, \qquad \text{for every} \quad x \in H.$$

We say that P_D is *isotone* if $x \leqslant y$ implies that $P_D(x) \leqslant P_D(y)$ for all x and y in H. When $D = K$ and P_K is isotone we say that K is an *isotone projection cone*. The isotone projection cones were studied in [9–14] and recently in [4]. In [9–14] several applications are also presented. It is interesting to know under which conditions the operator P_D is isotone. This problem has recently been studied by Isac in [15]. In this study the following inequality plays a central role:

Let $(H, <,>, K)$ be an ordered Hilbert space which is a vectorial lattice. We say that K satisfies *the property of four elements* (PFE) if, for every $x_1, x_2, x_3, x_4 \in H$ such that $x_1 \geqslant x_3$, we have

$$(*) \quad \|x_1 - x_2\|^2 + \|x_3 - x_4\|^2 \geqslant \|x_1 - x_2 \vee x_4\|^2 + \|x_3 - x_2 \wedge x_4\|^2.$$

We say that a subset D of H is *latticially closed* if, for every $x, y \in D$, $x \vee y \in D$, and $x \wedge y \in D$. The following two results are proved in [15] by using the property (PFE):

THEOREM A. *If $(H, <,>, K)$ is an ordered Hilbert space such that with respect to the ordering defined by K it is a vectorial lattice, then K satisfies the property (PFE) if and only if $(H, <,>, K)$ is a Hilbert lattice.*

THEOREM B. *If $(H, <,>, K)$ is a Hilbert lattice, then, for every convex and latticially closed set D, the projection operator P_D is isotone.*

Now, in this paper we are interested to find the analogs of the property (PFE) in some Banach spaces. In particular, we will prove that a corresponding property $(\text{PFE})_p$ holds for cones in the spaces $L_p = L_p(\Omega, \mu)$, $1 \leqslant p \leqslant \infty$ (see Corollary 2.3). Here, and in the sequel, (Ω, μ) denotes a σ-finite measure space. In fact, in this case the inequality corresponding to (PFE) can be sharpened in a way so that it even holds in the reversed direction for $0 < p \leqslant 1$. We even state our main result in Section 2 (Theorem 2.2) in a setting where the L_p-spaces are replaced by the more general functional spaces A_p. Moreover, the connection to Clarkson type inequalities and interpolation is discussed, the limiting inequalities are described and some further generalizations are pointed out (see Section 3). The property $(\text{PFE})_V$ for Lyapunov functionals is introduced and studied (see Section 4) and some applications are presented (see Section 5).

2. ON THE PROPERTY (PFE) IN SOME FUNCTIONAL SPACES

Let F denote a nonempty set and let L denote an additive set of real valued functions $g: F \to \mathbb{R}$ (i.e., if $f, g \in L$, then $f+g \in L$). we also consider an isotone additive functional $A: L \to \mathbb{R}$, i.e., A satisfies

(i) $A(f+g) = A(f) + A(g)$ for all $f, g \in L$,

(ii) $f, g \in L, f(t) \leqslant g(t)$ for all $t \in F \Rightarrow A(f) \leqslant A(g)$.

If $0 < p < \infty$ we say that $f \in A_p$ if $A_p(f) = (A(|f|^p))^{1/p} < \infty$.

The key step in our investigation is to prove the following lemma of independent interest:

LEMMA 2.1. *Let x_1, x_2, x_3, and x_4 be real numbers such that $x_1 \geqslant x_3$. If $p \geqslant 1$, then $(\theta)_p$: $|x_1 - x_2|^p + |x_3 - x_4|^p \geqslant |x_1 - x_2 \vee x_4|^p + |x_3 - x_2 \wedge x_4|^p + 2\,|(x_4 \wedge x_1 - x_3 \vee x_2) \vee 0|^p$. The inequality $(\theta)_p$ holds in the reversed direction if $0 < p \leqslant 1$.*

Proof. First we remark that if $x_2 \geqslant x_4$, then the third term on the right-hand side in $(\theta)_p$ is equal to zero and $(\theta)_p$ reduces to an equality. Hence we may without loss of generality assume that $x_2 \leqslant x_4$.

Let $p \geqslant 1$. We need to consider the following six cases:

(1) $x_3 \leqslant x_1 \leqslant x_2 \leqslant x_4$. Put $a = x_1 - x_3$, $b = x_2 - x_1$, $c = x_4 - x_2$ and consider the function $h(a) = (a+b+c)^p - (a+b)^p$.

We note that h is a nondecreasing function so that, in particular, $h(a) \geqslant h(0)$, which implies that

$$(\theta')_p\colon b^p + (a+b+c)^p \geqslant (a+b)^p + (b+c)^p,$$

i.e.,

$$(x_2 - x_1)^p + (x_4 - x_3)^p \geqslant (x_4 - x_1)^p + (x_2 - x_3)^p,$$

and this is the inequality $(\theta)_p$ for the case at hand.

(2) $x_2 \leqslant x_4 \leqslant x_3 \leqslant x_1$. Put $a = x_4 - x_2$, $b = x_3 - x_4$, $c = x_1 - x_3$, and argue as in (1) to find that

$$(x_1 - x_2)^p + (x_3 - x_4)^p \geqslant (x_1 - x_4)^p + (x_3 - x_2)^p,$$

via $(\theta')_p$ and $(\theta)_p$ is proved also for this case.

(3) $x_3 \leqslant x_2 \leqslant x_1 \leqslant x_4$. Put $a = x_2 - x_3$, $b = x_1 - x_2$, and $c = x_4 - x_1$. Then, by using the fact that $(x+y)^p \geqslant x^p + y^p$ for all $x, y \geqslant 0$, we obtain

$$(x_1 - x_2)^p + (x_4 - x_3)^p = b^p + (a+b+c)^p \geqslant a^p + c^p + 2b^2$$
$$= (x_4 - x_1)^p + (x_2 - x_3)^p + 2(x_1 - x_2)^p,$$

and $(\theta)_p$ is again proved for the actual case.

(4) $x_2 \leqslant x_3 \leqslant x_4 \leqslant x_1$. The proof is similar as that of the proof of the case (3).

(5) $x_2 \leqslant x_3 \leqslant x_1 \leqslant x_4$. Put $a = x_3 - x_2$, $b = x_1 - x_3$ and $c = x_4 - x_1$. Then

$$(x_1 - x_2)^p + (x_4 - x_3)^p = (a+b)^p + (b+c)^p \geqslant a^p + c^p + 2b^p$$
$$= (x_4 - x_1)^p + (x_3 - x_2)^p + 2(x_1 - x_3)^p,$$

and the inequality $(\theta)_p$ is proved for this case too.

(6) $x_3 \leqslant x_2 \leqslant x_4 \leqslant x_1$. The proof is analogous to that of the case (5).

The proof of the case $0 < p \leqslant 1$ only consists of minor modifications of the proof above. ∎

Now we are ready to formulate our main result in this section.

THEOREM 2.2. *Let A be an isotone additive functional and let $1 \leqslant p < \infty$. If $f_1, f_2, f_3, f_4 \in A_p$ and $f_1 \geqslant f_3$, then*

$$(\beta)_p\colon A_p^p(f_1 - f_2) + A_p^p(f_3 - f_4)$$
$$\geqslant A_p^p(f_1 - f_2 \vee f_4)$$
$$+ A_p^p(f_3 - f_2 \wedge f_4) + 2A_p^p((f_4 \wedge f_1 - f_3 \vee f_2) \vee 0).$$

If $0 < p \leqslant 1$, then $(\beta)_p$ holds in the reversed direction.

Proof. Apply the functional A to the functions

$$f(t) = |f_1(t) - f_2(t)|^p + |f_3(t) - f_4(t)|^p,$$
$$g(t) = |f_1(t) - f_2(t) \vee f_4(t)|^p + |f_3(t) - f_2(t) \wedge f_4(t)|^p$$
$$+ 2|(f_4(t) \wedge f_1(t) - f_3(t) \wedge f_2(t)) \vee 0|^p,$$

and use Lemma 2.1 together with the additivity and isotonicity properties of A (note that $A(|x|^p) = A_p^p(x)$). ∎

Corollary 2.3. *If $f_1, f_2, f_3, f_4 \in L_p(\Omega, \mu)$, $1 < p < \infty$, and $f_1(x) \geqslant f_3(x)$ a.e., then*

$$\|f_1 - f_2\|_p^p + \|f_3 - f_4\|_p^p \geqslant \|f_1 - f_2 \vee f_4\|_p^p + \|f_3 - f_2 \wedge f_4\|_p^p + 2\|(f_4 \wedge f_1 - f_3 \vee f_2) \vee 0\|_p^p.$$

For the case $0 < p < 1$ the inequality holds in the reversed direction and for the case $p = 1$ we have equality.

Proof. We apply Theorem 2.2 with $A(f) = \int_\Omega f \, d\mu$. ▮

Example 2.4. For $L_p = L_p(\Omega, \mu)$ with $1 \leqslant p < \infty$ the cone $K = \{f \in L_p(\Omega, \mu) \mid f \geqslant 0 \text{ a.e.}\}$ has the following property:

$$(\text{PFE})_p \begin{cases} \text{if } f_1, f_2, f_3 \text{ and } f_4 \in L_p, \text{ then} \\ \|f_1 - f_2\|_p^p + \|f_3 - f_4\|_p^p \geqslant \|f_1 - f_2 \vee f_4\|_p^p + \|f_3 - f_2 \wedge f_4\|_p^p. \end{cases}$$

We note that $(\text{PFE})_2$ coincides with (PFE) for $H = L_2(\Omega, \mu)$ (and $x_i = f_i$). Moreover, Corollary 2.3 shows that in this case (PFE) holds even in the sharper form obtained by adding the term $2\|(x_4 \wedge x_1 - x_3 \vee x_2) \vee 0\|^2$ to the right hand side of (*).

3. FURTHER REMARKS AND RESULTS

Remark 3.1. By taking both sides in $(\theta)_p$ to the power $1/p$ and letting $p \to \infty$ we obtain the following limiting inequality of the inequality $(\theta)_p$:

$$(\theta)_\infty\colon |x_1 - x_2| \vee |x_3 - x_4| \geqslant |x_1 - x_2 \vee x_4| \vee |x_3 - x_2 \wedge x_4| \vee |(x_4 \wedge x_1 - x_3 \vee x_2) \vee 0|.$$

Similarly, for the case when $x_4 \wedge x_1 - x_3 \vee x_2 \leqslant 0$ we can let $p \to 0+$ to obtain the following limiting inequality of the reversed inequality $(\theta)_p$:

$$(\theta)_0\colon |x_1 - x_2|\,|x_3 - x_4| \leqslant |x_1 - x_2 \vee x_4|\,|x_3 - x_2 \wedge x_4|.$$

Here we only use the fact that the scale of power means $\{P_\alpha\}$ converges to the geometric mean P_0 as $\alpha \to 0+$.

The corresponding limiting inequalities β_0 and β_∞ hold too.

Remark 3.2. Another proof of the inequality $(\theta)_p$, $1 < p < \infty$, can be obtained by interpolating with the complex method (c.f. [16]) between the inequality $(\theta)_\infty$ and the easily verified equality

$$(\theta)_1\colon |x_1 - x_2| + |x_3 - x_4| = |x_1 - x_2 \vee x_4| + |x_3 - x_2 \wedge x_4| + 2|(x_4 \wedge x_1 - x_3 \vee x_2) \vee 0|.$$

Remark 3.3. An inequality of the type $(\theta)_p$ (or $(\beta)_p$) with four involved numbers (functions) can also be obtained by applying suitable variants of Clarkson's inequality. In particular, by using [16, Proposition 2.1] we find that

$$(\phi)_p\colon |x_1-x_2|^p+|x_3-x_4|^p \geqslant \min(2^{-1}, 2^{1-p})(|x_1+x_3-x_2-x_4|^p+|x_1+x_4-x_2-x_3|^p)$$

for every p, $0<p<\infty$. Both of the inequalities $(\theta)_p$ and $(\phi)_p$ are sharp but the cases of equality are different.

If in the definition of the (isotone) linear functional A the condition (i) is replaced by

$$\text{(i)}' \quad A(f+g)\leqslant A(f)+A(g)[A(f+g)\geqslant A(f)+A(g)] \qquad \text{for all } f, g\in L,$$

then we say that A is an (isotone) *subadditive* [*superadditive*] *functional.*

Remark 3.4. Assume that the assumptions in Theorem 2.2 are satisfied except that A is only *subadditive*. Then our method of proof only gives the weaker inequality

$$(\beta')_p: A_p^p(f_1-f_2)+A_p^p(f_3-f_4) \geqslant A_p^p((|f_1-f_2\vee f_4|^p+|f_3-f_2\wedge f_4|^p+2|(f_4\wedge f_1-f_3\vee f_2)\vee 0|^p)^{1/p}),$$

$(p\geqslant 1)$ instead of $(\beta)_p$. Similarly, for the superadditive case we can only get the inequality

$$(\beta'')_p\colon A_p^p((|f_1-f_2|^p+|f_3-f_4|^{1/p}) \geqslant A_p^p(f_1-f_2\vee f_4)+A_p^p(f_3-f_2\wedge f_4)+2A_p^p((f_4\wedge f_1-f_3\vee f_2)\vee 0).$$

Remark 3.5. One typical isotone additive functional is the L_1-functional $A(f)=\int_\Omega f\,d\mu$ considered in Corollary 2.3 which, via the A_p-construction (convexification) defines the usual spaces $L_p(\Omega,\mu)$. Unfortunately, the functionals A corresponding to the other "usual" function spaces (e.g. Orlicz spaces, Lorentz spaces, Sobolev spaces, Besov spaces, X^p spaces, etc.) are not additive but only subadditive and therefore our technique only give the somewhat weaker inequality $(\beta')_p$.

EXAMPLE 3.6. Let $X=(X, \|\cdot\|_X)$ be any Banach function space. Then the space X^r, $1\leqslant r<\infty$, is defined by the norm $\|f\|_{X^r}=(\||f|^r\|_X)^{1/r}$. It is well-known that the functional $A(f)=\|f\|_{X^r}$ is subadditive (see e.g. [19, Lemma 1.2]) and, thus, according to Remark 3.4, it holds that *if* $f_1, f_2, f_3, f_4 \in X^q$ *and* $f_1 \geqslant f_3$, *then*

$$(\| f_1 - f_2 \|_{X^q})^p + (\| f_3 - f_4 \|_{X^q})^p$$
$$\geqslant (\| (|f_1 - f_2 \vee f_4|^p + |f_3 - f_2 \wedge f_4|^p + 2|(f_4 \wedge f_1 - f_3 \vee f_2) \vee 0)|^p)^{1/p} \|_{X^q})^p$$

for any $q \geqslant p \geqslant 1$.

For the case $q = p \geqslant 1$ and $X = L_1(\Omega, \mu)$ this is exactly the statement in Corollary 2.3.

Remark 3.7. By using the arguments applied in [16] we can obtain further formal generalizations of our results. Here we only mention the following result:

THEOREM 2.2′. *Let* A, f_1, f_2, f_3, *and* f_4 *be as in* Theorem 2.2. *If* $p \geqslant 1$, *then*

$$(\beta)_{p,r}: (A_p^r(f_1 - f_2) + A_p^r(f_3 - f_4))^{1/r}$$
$$\geqslant K_{r,p}(A_p^r(f_1 - f_2 \vee f_4) + A_p^r(f_3 - f_2 \wedge f_4)$$
$$+ 2A_p^r((f_4 \wedge f_1 - f_3 \vee f_2) \vee 0))^{1/r},$$

where $K_{r,p} = 2^{1/r - 1/p}$ *when* $p \leqslant r$ *and* $K_{r,p} = 4^{1/p - 1/r}$ *when* $p \geqslant r$.

Proof. Put $a = A_p(f_1 - f_2)$, $b = A_p(f_3 - f_4)$, $c = A_p(f_1 - f_2 \vee f_4)$, $d = A_p(f_3 - f_2 \wedge f_4)$ and $e = A_p((f_4 \wedge f_1 - f_3 \vee f_2) \vee 0)$ and consider the functions $g(r) = (a^r + b^r)^{1/r}$ and $h(r) = (c^r + d^r + 2e^r)^{1/r}$. The imbedding between 1_p spaces shows that both g and h are nonincreasing functions of r and the monotonicity of the scale of power means guarantees that the functions $g(r)\,2^{-1/r}$ and $h(r)\,4^{-1/r}$ both are nondecreasing functions of r. Moreover, by Theorem 2.2, $g(p) \geqslant h(p)$, and we find that if $p \leqslant r$, then

$$(A_p^r(f_1 - f_2) + A_p^r(f_3 - f_4))^{1/r}$$
$$= (a^r + b^r)^{1/r} = 2^{1/r} g(r)\,2^{-1/r} \geqslant 2^{1/r} g(p) 2^{-1/p}$$
$$= K_{r,p}\, g(p) \geqslant K_{r,p}\, h(p) \geqslant K_{r,p}\, h(r) = K_{r,p}(c^r + d^r + 2e^r)^{1/r}$$
$$= K_{r,p}(A_p^r(f_1 - f_2 \vee f_4) + A_p^r(f_3 - f_2 \wedge f_4)$$
$$+ 2A_p^r((f_4 \wedge f_1 - f_3 \vee f_2) \vee 0))^{1/r}.$$

The proof of the case $p \geqslant r$ is similar. ∎

A similar formal generalization can be given for the case $0 < p \leqslant 1$ and Theorem 2.2 coincides with the special case $r = p$.

Moreover, the statements above can be further generalized by using the well-known fact that also the more general scale of Gini means $G(\alpha, \beta)$ increases in both of the variables α and β (note that $G(\alpha, 0) = P(\alpha)$), see [19] and c.f. concluding remark 3 in [16].

If $1 < p < \infty$ the space $L_p(\Omega, \mu)$ is a uniformly convex Banach space (see [5]), ordered by the closed convex cone $K = \{f \in L_p(\Omega, \mu) \mid f \geqslant 0 \text{ a.e.}\}$ and with respect to this ordering it is a vectorial lattice.

If $D \subset L_p(\Omega, \mu)$ is any closed convex cone the projection operator P_D is well defined, i.e., for every $x \in L_p(\Omega, \mu)$, $P_D(x)$ is a singleton (see [3], Proposition 5, p.194). By using the property $(\text{PFE})_p$ proved before we obtain the following result:

THEOREM 3.8. *For every latticially closed and closed convex set* $D \subset L_p(\Omega, \mu)$, $1 < p < \infty$, *the projection operator* P_D *is isotone with respect to the ordering defined by K.*

Proof. See [15]. ∎

4. ON THE PROPERTY $(\text{PFE})_V$ FOR LYAPUNOV FUNCTIONALS

First we recall that the formula used in the definition of the metric projection operator in a Hilbert space H is equivalent to the minimization problem

$$(\gamma): \quad P_D(x) = \bar{x};\ \|x - \bar{x}\|^2 = \inf_{\xi \in D} \|x - \xi\|^2.$$

Here $D \subset H$ is a closed convex set. We also remark that $V(x, \xi) = \|x - \xi\|^2$ can be considered not only as the square of the distance between the points x and ξ but also as the Lyapunov functional with respect to ξ with x fixed. Hence, we can rewrite (γ) in the form

$$(\gamma_1): \quad P_D(x) = \bar{x};\ V(x, \bar{x}) = \inf_{\xi \in D} V(x, \xi).$$

Since in Hilbert spaces

$$V(x, \xi) = \|x\|^2 - 2\langle x, \xi \rangle + \|\xi\|^2$$

it is natural to consider the following construction proposed by Alber [1]:

Let E be a uniformly convex and uniformly smooth Banach space and let $J\colon E \to E^*$ be the duality mapping. We consider the Lyapunov functional

$$V(J(x), \xi) = \|J(x)\|_{E^*}^2 - 2\langle J(x), \xi \rangle + \|\xi\|_E^2$$

and we define

$$(\gamma_2):\quad P_D^V(x)=x_*;\ V(J(x),x_*)=\inf_{\xi\in D} V(J(x),\xi),$$

where $D\subset E$ is a closed convex set.

The function V has some good properties as for example:

(i) $V(J(x),\xi)\geqslant 0$ for all $x,\xi\in E$,

(ii) $V(J(x),\xi)\to\infty$ if $\|x\|\to\infty$ (or $\|\xi\|\to\infty$).

Because of these properties we see that $P_D^V(x)$ is well defined and it is unique. If $(E,\|\,\|,K)$ is an ordered Banach space which is uniformly convex, uniformly smooth and a vectorial lattice we introduce the following definition:

Definition 4.1. We say that the cone K satisfies the property of four elements, $(\mathrm{PFE})_V$, if $x_1,x_2,x_3,x_4\in E$ are such that $x_1\geqslant x_3$, then

$$V(J(x_1),x_2)+V(J(x_3),x_4)\geqslant V(J(x_1),x_2\vee x_4)+V(J(x_3),x_2\wedge x_4).$$

Our main result in this section reads:

Theorem 4.2. *Let $(E,\|\,\|,K)$ be an ordered Banach space which is uniformly convex and uniformly smooth. If the cone K satisfies the property $(\mathrm{PFE})_V$, then, for every latticially closed, closed convex set $D\subset E$, the projection operator P_D^V is isotone with respect to the order defined by K.*

Proof. Let $x,y\in E$ be such that $x\leqslant y$. We denote $x_1=y$, $x_2=P_D^V(y)$, $x_3=x$, and $x_4=P_D^V(x)$. Since K satisfies the property $(\mathrm{PFE})_V$ we have

$$\begin{aligned}&V(J(y),P_D^V(y))+V(J(x),P_D^V(x))\\&\quad\geqslant V(J(y),P_D^V(y)\vee P_D^V(x))+V(J(x),P_D^V(y)\wedge P_D^V(x)).\end{aligned}$$

Moreover, since D is latticially closed it follows from the definition of P_D^V that

$$V(J(y),P_D^V(y))\leqslant V(J(y),P_D^V(y)\vee P_D^V(x))\ \text{and}$$
$$V(J(x),P_D^V(x))\leqslant V(J(x),P_D^V(y)\wedge P_D^V(x)).$$

By combining these inequalities we obtain that

$$\begin{aligned}V(J(y),P_D^V(y))&\leqslant V(J(y),P_D^V(y)\vee P_D^V(x))\\&\leqslant V(J(y),P_D^V(y))+V(J(x),P_D^V(x))\\&\qquad-V(J(x),P_D^V(y)\wedge P_D^V(x))\\&\leqslant V(J(y),P_D^V(y)).\end{aligned}$$

Hence

$$V(J(y), P_D^V(y)) = V(J(y), P_D^V(y) \vee P_D^V(x)),$$

and from the uniqueness of $P_D^V(y)$ we deduce that $P_D^V(y) = P_D^V(y) \vee P_D^V(x)$, that is, $P_D^V(x) \leqslant P_D^V(y)$, which means that P_D^V is isotone.

Open Problem. It is important to characterize the cones which satisfies the property $(\mathrm{PFE})_V$.

5. APPLICATIONS

We indicate now some possible applications of the isotonicity of the projection operator onto a latticially closed set in a Hilbert lattice.

I. Let $A: H \to H$ be an arbitrary operator (not necessary linear), $\alpha \in \mathbb{R}_+ \setminus \{0\}$ an arbitrary number and $\varphi \in H$ a fixed element. Let $D \subset H$ be a closed convex and latticially closed set. We consider the following variational problem:

$$VI(A, D)\text{: find } x_* \in D \text{ such that } \langle A(x_*) - \varphi, y - x_* \rangle \geqslant 0 \text{ for all } y \in D.$$

II. Let $f: H \to R$ be a continuous differentiable function and $D \subset H$ a closed convex and latticially closed set. We denote by $\nabla f(x)$ the gradient of f at the point x. We say that $x_* \in D$ is a *constrained stationary point* (CSP) for f if and only if $x_* = P_D[x_* - \nabla f(x_*)]$. We consider now the problem

$$\mathrm{CSP}(f, D)\text{: find } x_* \in D \text{ such that } x_* = P_D[x_* - \nabla f(x_*)].$$

The existence and the approximation of the solution x_* of the problem $\mathrm{CSP}(f, D)$ have been considered by several authors such as McCormic and Tapia [17], Golomb and Tapia [7], and Phelps [20, 21].

In particular this problem is closed to the problem to minimize a continuous differentiable function with respect to an *order simplex* $\mathscr{S}_n(a, b)$ in R^n, where, by definition,

$$\mathscr{S}_n(a, b) = \{t = (t_1, t_2, ..., t_n) \in R^n \mid a \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n = b\}.$$

An interesting paper on this subject is [8]. In order to study the problems $VI(A, D)$ and $\mathrm{CSP}(f, D)$ we will use the isotonicity of the projection operator and the concept of *heterotonic operator* introduced by V. I. Opoitsev [18].

DEFINITION 5.1 [18]. We say that the mapping $T: H \to H$ is heterotonic on a set $D \subset H$ if and only if there exists a mapping $f_T: H \times H \to H$ such that

(1) $f_T(x, x) = T(x)$, *for all* $x \in D$,

(2) $f_T(x, y)$ *is monotone increasing with respect to* x *for all* y,

(3) $f_T(x, y)$ *is monotone decreasing with respect to* y *for all* x.

We remark that in Definition 5.1 the mapping f_T is *not* unique.

DEFINITION 5.2. If T is heterotonic we say that (x_*, y_*) is a coupled fixed point of T if $f_T(x_*, y_*) = x_*$, and $f_T(y_*, x_*) = y_*$.

Here we note that every fixed point is a coupled fixed point if we take (x_*, x_*). The converse is not true.

DEFINITION 5.3. We say that a coupled fixed point (x_*, y_*) of a heterotonic operator T is minimal and maximal with respect to D if for every coupled fixed point $(\bar{x}, \bar{y})$ of T in D we have $x_* \leqslant \bar{x} \leqslant y_*$ and $x_* \leqslant \bar{y} \leqslant y_*$.

If $u_0, v_0 \in D$ are two elements such that $u_0 \leqslant v_0$, then we denote by $[u_0, v_0]_0$ the order interval defined by u_0 and v_0, that is, $[u_0, v_0]_0 = \{x \in H \mid u_0 \leqslant x \leqslant v_0\}$.

DEFINITION 5.4. We say that the order interval $[u_0, v_0]_0$ is strongly invariant for the heterotonic operator T if $u_0 \leqslant f_T(u_0, v_0)$ and $f_T(v_0, u_0) \leqslant v_0$.

We will also use the concepts of *nonexpansive operator* and *condensing operator* (in Sadovski's sense) as defined and studied in [22].

We are now ready to formulate our main result in this section.

THEOREM 5.5. *Let* $(H, <, >, K)$ *be a Hilbert lattice,* $D \subset H$ *a subset and* $T: H \to H$ *a heterotonic operator with respect to* D. *Assume that*

(1) f_T *is continuous,*

(2) *there exist* $x_0, y_0 \in D$ *such that* $[x_0, y_0]_0$ *is strongly invariant for* T *and* $[x_0, y_0]_0 \subset D$,

(3) T *is nonexpansive and condensing.*

Then T *has a coupled fixed point* (x_*, y_*) *which is minimal and maximal with respect to* $[x_0, y_0]_0$. *Moreover,* x_* *and* y_* *can be computed by iterations* *and the set of fixed points of T with respect to $[x_0, y_0]_0$ is nonempty and contained in $[x_*, y_*]_0$.*

Proof. Using x_0 and y_0 we define the sequences $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ by

$$\begin{cases} x_{n+1}=f_T(x_n, y_n) & \text{for all} \quad n\in N, \\ y_{n+1}=f_T(y_n, x_n) & \text{for all} \quad n\in N. \end{cases}$$

We have $x_0 \leqslant x_1 = f_T(x_0, y_0)$, $y_1 = f_T(y_0, x_0) \leqslant y_0$ and, by induction, we find that $x_n \leqslant x_{n+1} \leqslant y_{n+1} \leqslant y_n$ for all $n\in N$.

Since T is heterotonic we can show that $T([x_n, y_n]_0) \subseteq [x_n, y_n]_0$ for all $n\in N$. Moreover, because H is a Hilbert lattice, K is normal and regular we obtain that $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ are convergent. We denote

$$x_* = \lim_{n\to\infty} x_n \qquad \text{and} \qquad y_* = \lim_{n\to\infty} y_n.$$

We have $x_* \leqslant y_*$ and from the definition of $\{x_n\}$ and $\{y_n\}$ and the continuity of f_T we deduce that $x_* = f_T(x_*, y_*)$ and $y_* = f_T(y_*, x_*)$, which means that (x_*, y_*) is a coupled fixed point for T.

Let now $(\bar{x}, \bar{y})$ be another coupled fixed point of T in $[x_0, y_0]_0$. We can show, again by induction, that $x_n \leqslant \bar{x} \leqslant y_n$ and $x_n \leqslant \bar{y} \leqslant y_n$ for all $n\in N$ and by taking the limit we have $x_* \leqslant \bar{x}$, $\bar{y} \leqslant y_*$, that is, (x_*, y_*) is a minimal and maximal coupled fixed point for T with respect to $[x_0, y_0]_0$. Since we have that $T([x_*, y_*]_0) \subseteq [x_*, y_*]_0$ we apply Brouwer's or Sadovski's fixed point theorem and we have also proved the last part of the conclusion of the theorem. ∎

Remark 5.6. If, for every $x, y \in [x_0, y_0]_0$ such that $x \neq y$ we have $f_T(x, y) \neq x$ and $f_T(y, x) \neq y$, then it is easily seen that we must in fact have $x_* = y_*$ in Theorem 5.5 and hence we find that x_* is a fixed point of T in this case.

COROLLARY 5.7. *We consider the problem $VI(A, D)$. Assume that, in addition to the assumptions indicated in the definition of this problem, the following hold*:

(a) *the function $h(x) = x - Ax + \varphi$ is continuous and $h(x) = T_1(x) + T_2(x)$, where T_1 is increasing and T_2 is decreasing,*

(b) *there exists $x_0, y_0 \in D$ such that $x_0 \leqslant y_0$, $[x_0, y_0]_0 \subset D$, $x_0 \leqslant T_1(x_0) + T_2(y_0)$ and $T_1(y_0) + T_2(x_0) \leqslant y_0$,*

(c) *h is nonexpansive or compact.*

Then the problem VI(A, D) has a solution. Moreover, the solution set is a subset of the order interval $[x_*, y_*]_0$, *where*

$$\begin{cases} x_* = \lim_{n\to\infty} x_n,\ y_* = \lim_{n\to\infty} y_n, \\ x_{n+1} = P_D(T_1(x_n) + T_2(y_n)), & n \in N, \\ y_{n+1} = P_D(T_1(y_n) + T_2(x_n)), & n \in N. \end{cases}$$

Proof. We can apply Theorem 5.5 since h is heterotonic with $f_T(x, y) = P_D(T_1(x) + T_2(y))$ for all $x, y \in D$. ∎

COROLLARY 5.8. *We consider the problem* CSP(f, D). *Suppose that, in addition to the assumptions indicated in the definition of this problem, the following assumptions are satisfied*:

(a) *the function* $h(x) = x - \nabla f(x)$ *is continuous and there exist* T_1 *and* T_2 *such that* $h(x) = T_1(x) + T_2(x)$, *with* T_1 *increasing and* T_2 *decreasing,*

(b) *there exists* $x_0, y_0 \in D$ *such that* $x_0 \leqslant y_0$, $[x_0, y_0]_0 \in D$, $x_0 \leqslant T_1(x_0) + T_2(y_0)$ *and* $T_1(y_0) + T_2(x_0) \leqslant y_0$,

(c) *h is nonexpansive or compact.*

Then the function f has a constrained stationary point in D. Moreover, the order interval $[x_*, y_*]_0$ *contains all the constrained stationary points of f in* $[x_0, y_0]_0$, *where*

$$\begin{cases} x_* = \lim_{n\to\infty} x_n,\ y_* = \lim_{n\to\infty} y_n, \\ x_{n+1} = P_D(T_1(x_n) + T_2(y_n)), & n \in N, \\ y_{n+1} = P_D(T_1(y_n) + T_2(x_n)), & n \in N. \end{cases}$$

Proof. Apply Theorem 5.5 in the same way as in the proof of Corollary 5.7. ∎

Final Remarks 5.9. (1) By analyzing our proof of Theorem 5.5 we see that in this theorem and Corollaries 5.7–5.8 we can replace the assumption that h has a decomposition $h = T_1 + T_2$ by the assumption that "h is heterotonic."

(2) If $\dim H < \infty$ the assumption 3) in Theorem 5.5 and Corollaries 5.7–5.8 is not necessary since using the continuity of h, the convexity of $[x_*, y_*]_0$, the fact that this interval is bounded and-applying Brouwer's fixed point theorem we obtain the conclusions of the theorem.

(3) Concerning the applications presented before it is particularly interesting to extend Corollary 5.7 and Corollary 5.8 to Banach spaces by using our results presented in this paper and the results presented in [1]

and [14]. Evidently, to develop this idea it is necessary to first study the isotonicity of the projection operator with respect to a Lyapunov functional. We intend to present this development in a forthcoming paper.

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REFERENCES

1. Ya. I. Alber, Generalized projection operators in Banach spaces: Properties and applications, preprint, Department of Mathematics, Technion—Israel Institute of Technology, 1993.
2. C. Baiocchi and A. Capelo, "Variational and Quasivariational Inequalities. Application to Free-Boundary Problems," Wiley, New York, 1984.
3. B. Beauzamy, "Introduction to Banach Spaces and Their Geometry," North-Holland, Amsterdam/New York/Oxford, 1985.
4. S. J. Bernau, Isotone projection cones, preprint, Dept. Math. Sciences, The Univ. of Texas at El Paso, 1993.
5. J. A. Clarkson, Uniformly convex spaces, *Trans. Amer. Math. Soc.* **40** (1936), 396–414.
6. Fr. Deutsch, The method of alternating orthogonal projections, *in* "Approximation Theory, Spline Functions and Applications" (S.P. Singh, Ed.), pp. 105–121, Kluwer, Dordrecht, 1992.
7. M. Golomb and R. A. Tapia, The metric gradient in normed linear spaces, *Numer. Math.* **20** (1972), 115–124.
8. S. J. Gratzinger and C. Witzgall, Projections onto order simplexes, *Appl. Math. Optim.* **12** (1984), 247–270.
9. G. Isac and A. B. Németh, Monotonicity of metric projections onto positive cones in order Euclidean spaces, *Ark. Math.* **46** (1986), 568–576.
10. G. Isac and A. B. Németh, Every generating isotone projection cone is latticial and correct, *J. Math. Anal. Appl.* **147** (1990), 53–62.
11. G. Isac and A. B. Németh, Isotone projection cones in Hilbert spaces and the complementarity problem, *Boll. Un. Math. Ital.* **7** (1990), 773–802.
12. G. Isac and A. B. Németh, Projection methods, isotone projection cones and the complementarity problem, *J. Math. Anal. Appl.* **153** (1990), 258–275.
13. G. Isac and A. B. Németh, Isotone projection cones in Euclidean spaces, *Ann. Sci. Math. Québec* **16** (1992), 35–52.
14. G. Isac, Iterative methods for the General Order Complementarity Problem, *in* "Approximation Theory, Spline Functions and Applications" (S.P. Singh, Ed.), pp. 365–380, Kluwer, Dordrecht, 1992.
15. G. Isac, On the order monotonicity of the metric projection operator, *in* "Approximation Theory, Wavelets and Applications" (S.P. Singh, Ed.), pp. 365–379, Kluwer, Dordrecht, 1995.
16. L. Maligranda and L. E. Persson, On Clarkson's inequalities and interpolation, *Math. Nachr.* **155** (1992), 187–197.

17. G. P. McCormick and R. A. Tapia, The gradient projection method under mild differentiability conditions, *SIAM J. Control* **10** (1972), 93–98.
18. V. I. Opoitsev, A generalization of the theory of monotone and concave operators, *Transl. Moscow Math. Soc.* **2** (1979), 243–279.
19. J. Peetre and L. E. Persson, A general Beckenbach's inequality with applications, *in* "Function Spaces, Differential Operators and Nonlinear Analysis" (L. Päivärinta, Ed.), Pitman Research Notes in Math., Vol. 211, pp. 125–139, Pitman, London/Boston, 1989.
20. R. R. Phelps, Metric projection and gradient projection method in Banach spaces, *SIAM J. Control Optim.* **23** (1985), 973–977.
21. R. R. Phelps, The gradient projection method using Curry's step-length, *SIAM J. Control Optim.* **24** (1986), 692–699.
22. E. Zeidler, "Nonlinear Functional Analysis and Its Applications," Vol. I, Springer-Verlag, New York/Berlin/Heidelberg, 1985.